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Estimating returns to scale in the presence of undesirable factors in data envelopment analysis

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Abstract

This research identifies returns to scale (RTS) of efficient decision making units (DMUs) with desirable (good) and undesirable (bad) inputs and outputs by presenting a new DEA (data envelopment analysis) approach. In this study, we first introduce a new input-output oriented model to determine efficient DMUs in the presence of undesirable factors and then, returns to scale of these DMUs are estimated by presenting a new non-radial DEA model.

So far several RTS approaches has been proposed in DEA literature by many researchers, such as Banker and Thrall's, Golany and Yu's, Khodabakhshi's et al., and Eslami and Khoveyni's RTS approaches. In the proposed approaches, all inputs and outputs are respectively considered as desirable inputs and outputs while in real world, both desirable and undesirable data may be present. Note that advantage of our proposed approach is capable of estimating RTS of efficient DMUs in the presence of desirable and undesirable data. It is noticeable that, since an inefficient decision making unit (DMU) has more than one projection on the empirical function thus different returns to scales can be obtained for projections of the inefficient DMU by using our proposed RTS approach.

Lastly, an empirical example for illustrating purpose is presented and also directions for future research are suggested.

Keywords: Data Envelopment Analysis (DEA), Returns to Scale (RTS), Efficiency, Undesirable Factors

1. Introduction

Data envelopment analysis (DEA) applies linear programming (LP) problems to assess the relative efficiencies and inefficiencies of decision making units (DMUs) with multiple inputs and outputs. Once the efficient frontier is identified by DEA and also using DEA, the performance of inefficient DMUs is

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improved by decreasing and increasing their inputs and outputs, respectively. However, in real world, desirable (good) and undesirable (bad) input and output factors may be present. For instance, in a cement production factory, one of the undesirable outputs of pollutions is smoke. If inefficiency exists in the production, the undesirable pollutions should be decreased for improving the inefficiency. On the other word, in order to assess the production performance of cement factory, desirable and undesirable outputs should be treated differently.

Therefore for improving the performance of an inefficient DMU, desirable and undesirable outputs should be respectively increasing and decreasing and also desirable and undesirable inputs should be decreasing and increasing, respectively. However, in the standard DEA models, inputs and outputs are only decreasing and increasing, respectively, and increasing inputs and decreasing outputs are not allowed in these models. A non-linear DEA program was developed by Färe et al. [1] for modeling the paper production system which desirable and undesirable outputs are increasing and decreasing, respectively. Furthermore, a DEA model was proposed by Vencheh et al. [2] for measuring efficiency of DMUs in the presence of undesirable factors. In addition, Amirteimoori et al. [3] presented a DEA model to improve the relative performance via decreasing undesirable outputs and increasing undesirable inputs.

In DEA literature, so far several approaches were presented for estimating returns to scale of DMUs with desirable inputs and outputs [4]. For instance, Banker [5] estimated most productive scale size (MPSS) by using DEA. Moreover, Seiford and Zhu [6] developed an alternative approach that preserve the linearly and convexity in Banker's et al. model [7]. Also, Banker et al. [7] provided an approach based on supporting hyperplane. In this vein, an alternative approach was provided to estimate RTS by Färe and Grosskopf [8] which is based on optimal solutions of BCC, CCR [9], and CCR-BCC models. In addition, a fractional model was provided to estimate MPSS by Cooper et al. [10]. A DEA method was introduced by Banker and Thrall [11] for estimating RTS of BCC-efficient DMUs. Also, Golany and Yu [12] presented another method to identify right and left returns to scales in DEA. In addition, Sueyoshi and Sekitani [13] introduced an alternative approach based on a non-radial (RAM) model. Moreover, Khodabakhshi et al. [14] presented an additive model approach for estimating returns to scale in imprecise data envelopment analysis. Also, Eslami et al. [15] introduced an imprecise-chance constrained input-output orientation model to estimate most productivity scale size (MPSS) in DEA. More recently, Eslami and Khoveyni [16] presented a DEA approach for estimating types and measuring values of right and left returns to scales of efficient DMUs.

In this paper, we first introduce a new input-output oriented DEA model for identifying efficient DMUs and then a non-radial model is presented to estimate returns to scale of efficient DMUs in DEA.

It is necessary to mention that the advantage of the proposed RTS approach is capable of estimating returns to scale of efficient DMUs in the presence of desirable (good) and undesirable (bad) input and output factors while the previous presented RTS approaches are incapable of estimating returns to scale in the presence undesirable data.

It is noteworthy that, since an inefficient DMU has more than one projection on the empirical function hence, different returns to scales can be obtained for projections of the inefficient DMU by using the proposed approach.

The remainder structure of this paper is organized as follows. Section 2 briefly explains some RTS approaches and related DEA models. In Section 3, our proposed RTS approach is described by presenting some theorems and models. An empirical example and computational results are provided to highlight the proposed approach in Section 4. Lastly, Section 5 includes concluding remarks along with future research agendas.

2. Preliminaries

In this section, we briefly explain describe some RTS approaches which are presented by Banker and Thrall [11], Khodabakhshi et al. [14], and Eslami and Khoveyni [16]. Furthermore, in order to facilitate our extension, some related DEA models are described as follows.

Now suppose a set of n DMUs, i.e. $\{DMU_j \mid j = 1,2,\ldots,n\}$, where each DMU_j produces s different outputs $y_{rj} \geq 0$ $(r = 1,2,\ldots,s)$ by using by m different inputs $x_{ij} \geq 0$ $(i = 1,2,\ldots,m)$ that $\mathbf{X}_j = (x_{1j},\ldots,x_{ij},\ldots,x_{mj}) \neq \mathbf{0}$ and $\mathbf{Y}_j = (y_{1j},\ldots,y_{rj},\ldots,y_{sj}) \neq \mathbf{0}$. Moreover, production possibility set (PPS) is defined as $PPS = \{(\mathbf{X},\mathbf{Y}) \mid \mathbf{Y}\ \textit{can be produced by}\ \mathbf{X}\}$. Production possibility set under variable RTS assumption is as below:

$$PPS_{BCC} = T_V = \left\{ (\mathbf{X},\mathbf{Y}) \middle| \sum_{j=1}^{n} \lambda_j \mathbf{X}_j \leq \mathbf{X}, \sum_{j=1}^{n} \lambda_j \mathbf{Y}_j \geq \mathbf{Y}, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0;\ j = 1,2,\ldots,n \right\}. \quad (1)$$

Note that in this study, "$*$" represents optimal solution values.

The input and output efficiency scores of a DMU under evaluation $(DMU_p; p \in \{1,2,\ldots,n\})$ can be evaluated by the following input and output oriented BCC models [7], respectively.

$$\begin{aligned}
Input-orientation: \theta^* = Min \quad & \theta - \varepsilon \left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \right) \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{ip}, \quad i = 1,\ldots,m, \qquad (2) \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rp}, \quad r = 1,\ldots,s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0, \quad j = 1,\ldots,n, \\
& s_i^- \geq 0, \quad i = 1,\ldots,m, \\
& s_r^+ \geq 0, \quad r = 1,\ldots,s,
\end{aligned}$$

$$\begin{aligned} Output-orientation:\ \varphi^* = Max \quad & \varphi + \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \\ s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{ip}, \quad i = 1,\ldots,m, \qquad (3) \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = \varphi y_{rp}, \quad r = 1,\ldots,s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \lambda_j \geq 0, \quad j = 1,\ldots,n, \\ & s_i^- \geq 0, \quad i = 1,\ldots,m, \\ & s_r^+ \geq 0, \quad r = 1,\ldots,s, \end{aligned}$$

where x_{ip} and y_{rp} represent the amounts of i^{th} input and r^{th} output for DMU_p, respectively. Also, ε is a non-Archimedean small positive number. Furthermore, input and output slacks are respectively presented by s_i^- $(i = 1, 2, \ldots, m)$ and s_r^+ $(r = 1, 2, \ldots, s)$.

Definition 1 (BCC-efficient). DMU_p is called BCC-efficient if and only if an optimal solution $(\theta^*, \boldsymbol{\lambda}^*, \mathbf{S}^{-^*}, \mathbf{S}^{+^*})$ $((\varphi^*, \boldsymbol{\lambda}^*, \mathbf{S}^{-^*}, \mathbf{S}^{+^*}))$ obtained from model (2) ((3)) satisfies $\theta^* = 1$ $(\varphi^* = 1)$ and has no slack ($\mathbf{S}^{-^*} = \mathbf{0}$, $\mathbf{S}^{+^*} = \mathbf{0}$). Otherwise, DMU_p is BCC-inefficient.

2.1. Banker and Thrall's RTS approach

Banker and Thrall [11] presented a DEA approach to estimate returns to scale of BCC-efficient DMUs. In order to evaluate DMU_p $\left(p \in \{1, 2, \ldots, n\}\right)$, consider the following dual (multiplier) form associated with model (2):

$$\begin{aligned} Max \quad & \mathbf{U}^t \mathbf{Y}_p + u_p \\ s.t. \quad & \mathbf{U}^t \mathbf{Y}_j - \mathbf{V}^t \mathbf{X}_j + u_p \leq 0, \quad j = 1,\ldots,n, \qquad (4) \\ & \mathbf{V}^t \mathbf{X}_p = 1, \\ & \mathbf{U} \geq \mathbf{0}, \\ & \mathbf{V} \geq \mathbf{0}. \end{aligned}$$

Note that hereafter, the superscript "t " indicates a vector transpose.

Now assume that $(\mathbf{U}^*, \mathbf{V}^*, u_p^*)$ is an obtained optimal solution from model (4). Banker and Thrall [11] presented the following theorem for estimating RTS of BCC-efficient DMUs.

Theorem 1. *Suppose that $(\mathbf{X}_p, \mathbf{Y}_p)$ is a point on the BCC-efficient frontier. Then, the following conditions identify the situation for RTS at the point:*

(i) *Increasing RTS (IRS) prevail at $(\mathbf{X}_p, \mathbf{Y}_p)$ if and only if $u_p^* > 0$ for all optimal solutions of BCC model in multiplier form.*

(ii) *Decreasing RTS (DRS) prevail at $(\mathbf{X}_p, \mathbf{Y}_p)$ if and only if $u_p^* < 0$ for all optimal solutions of BCC model in multiplier form.*

(iii) *Constant RTS (CRS) prevail at $(\mathbf{X}_p, \mathbf{Y}_p)$ if and only if $u_p^* = 0$ for at least one optimal solution of BCC model in multiplier form.*

Proof. Refer to [11]. □

In the next subsection, Khodabakhshi's et al. RTS approach [14] is briefly described.

2.2. Khodabakhshi's et al. RTS approach

Khodabakhshi et al. [14] were provided a DEA approach to identify returns to scale of BCC-efficient DMUs as bellows.

Suppose that DMU_p $(p \in \{1, 2, \dots, n\})$ is a point on the BCC-efficient frontier and consider the following additive model that has been presented by Charnes et al. [17] to evaluate the target DMU (DMU_p):

$$
\begin{aligned}
Max \quad & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{ip}, \quad i = 1, \dots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rp}, \quad r = 1, \dots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \qquad (5) \\
& \lambda_j \geq 0, \quad j = 1, \dots, n, \\
& s_i^- \geq 0, \quad i = 1, \dots, m, \\
& s_r^+ \geq 0, \quad r = 1, \dots, s.
\end{aligned}
$$

Definition 2. DMU_p is called efficient if and only if the obtained optimal value of objective function from model (5) is zero.
Now according to (1), we have the following theorem presented by Kodabakhshi et al. [14]:

Theorem 2 [14]. *Suppose that* DMU_p *with input-output combination* $\left(\mathbf{X}_p, \mathbf{Y}_p\right)$ *is efficient. Therefore, we have:*

(i) There is $0 < \xi < 1$ *so that* $\left(\xi\mathbf{X}_p, \xi\mathbf{Y}_p\right) \in PPS$ *is inefficient if and only if* $\left(\mathbf{X}_p, \mathbf{Y}_p\right)$ *has DRS.*

(ii) There is $\xi > 1$ *so that* $\left(\xi\mathbf{X}_p, \xi\mathbf{Y}_p\right) \in PPS$ *is inefficient if and only if* $\left(\mathbf{X}_p, \mathbf{Y}_p\right)$ *has IRS.*

(iii) For each $\xi > 0$, $\left(\xi\mathbf{X}_p, \xi\mathbf{Y}_p\right) \in PPS$ *is efficient if and only if* $\left(\mathbf{X}_p, \mathbf{Y}_p\right)$ *has CRS.*

Proof. Refer to [14]. □

Now, the following model was proposed by Kodabakhshi et al. for estimating RTS of a DMU under evaluation $\left(DMU_p\right)$:

$$
\begin{aligned}
Max \quad & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \xi x_{ip}, \quad i = 1,\dots,m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = \xi y_{rp}, \quad r = 1,\dots,s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \qquad\qquad\qquad\qquad (6) \\
& \lambda_j \geq 0, \quad j = 1,\dots,n, \\
& s_i^- \geq 0, \quad i = 1,\dots,m, \\
& s_r^+ \geq 0, \quad r = 1,\dots,s.
\end{aligned}
$$

Now according to model (6), the RTS of DMU_p are detected as follows:

Theorem 3 [14]. *Suppose that* DMU_p *with input-output combination* $\left(\mathbf{X}_p, \mathbf{Y}_p\right)$ *is efficient. The following conditions estimate returns to scale of* DMU_p *being evaluated by model (10):*

(i) The optimal value of the objective function is greater than zero and $\xi^* > 1$ *if and only if* DMU_p *has IRS.*

(ii) The optimal value of the objective function is greater than zero and $\xi^* < 1$ *if and only if* DMU_p *has DRS.*

(iii) The optimal value of the objective function is zero if and only if DMU_p has CRS

Proof. Refer to [14]. □

In the next section, we will present our proposed approach for estimating RTS of efficient DMUs in the presence of undesirable factors.

3. New insights on estimating returns to scale in the presence of undesirable factors

Consider n DMUs, $\{DMU_j \mid j = 1, 2, \ldots, n\}$, with input-output combination $(\mathbf{X}_j, \mathbf{Y}_j) = (\mathbf{X}_j^g, \mathbf{X}_j^b, \mathbf{Y}_j^g, \mathbf{Y}_j^b) \in \mathbb{R}^{m+s}$. Note that $\mathbf{X}_j^g = (x_{1j}^g, \ldots, x_{ij}^g, \ldots, x_{m_1 j}^g) \in \mathbb{R}^{m_1}$ and $\mathbf{X}_j^b = (x_{1j}^b, \ldots, x_{ij}^b, \ldots, x_{m_2 j}^b) \in \mathbb{R}^{m_2}$ are desirable (good) and undesirable (bad) input vectors of DMU_j, respectively. Also, $\mathbf{Y}_j^g = (y_{1j}^g, \ldots, y_{rj}^g, \ldots, y_{s_1 j}^g) \in \mathbb{R}^{s_1}$ and $\mathbf{Y}_j^b = (y_{1j}^b, \ldots, y_{rj}^b, \ldots, y_{s_2 j}^b) \in \mathbb{R}^{s_2}$ are respectively desirable (good) and undesirable (bad) output vectors of DMU_j. It is noteworthy that $\mathbf{X}_j = (\mathbf{X}_j^g, \mathbf{X}_j^b) \underset{\neq}{\geq} \mathbf{0}$ and $\mathbf{Y}_j = (\mathbf{Y}_j^g, \mathbf{Y}_j^b) \underset{\neq}{\geq} \mathbf{0}$, therefore $m = m_1 + m_2$ and $s = s_1 + s_2$.

In what follows, we first introduce a new DEA model in input-output orientation for determining efficient DMUs in the presence undesirable factors and then, a new non-radial model is presented to estimate RTS of these efficient DMUs in DEA.

3.1. Determining efficient DMUs in the presence of undesirable inputs and outputs

In order to evaluate the efficiency of a target DMU $(DMU_p\ ;\ p \in \{1, 2, \ldots, n\})$ in the presence of undesirable data, we present the following input-output oriented DEA model:

$$
\begin{aligned}
Max \quad & z_p = \varphi_p - \theta_p \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^{g} \le \theta_p x_{ip}^{g}, \quad i = 1,\ldots,m_1, \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^{b} \le x_{ip}^{b}, \quad i = 1,\ldots,m_2, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{g} \ge \varphi_p y_{rp}^{g}, \quad r = 1,\ldots,s_1, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{b} \ge y_{rp}^{b}, \quad r = 1,\ldots,s_2, \qquad (7) \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \theta_p \le 1, \\
& \varphi_p \ge 1, \\
& \lambda_j \ge 0, \quad j = 1,\ldots,n.
\end{aligned}
$$

Note that in model (7), in order to improve the performance of DMU_p, desirable inputs and outputs of DMU_p are respectively decreased and increased while undesirable inputs and outputs of DMU_p are not allowed decreasing and increasing, respectively.

It is noticeable that $\theta_p = 1$, $\varphi_p = 1$, $\lambda_p = 1$, and $\lambda_j = 0$ $(j = 1,\ldots,n \ ; \ j \ne p)$ is a feasible solution of model (7) thus, model (7) is a feasible model.

Now, assume that $\left(z_p^*, \theta_p^*, \varphi_p^*, \boldsymbol{\lambda}^*\right)$ is an obtained optimal solution of model (7). Since $z_p = \varphi_p - \theta_p = 0$ and model (7) is as maximization model, therefore $z_p^* \ge 0$. Also according to desirable input constraints, θ_p^* is positive.

Definition 3. DMU_p is called efficient under model (7) if and only if two following conditions are satisfies:

(i) $z_p^* = 0$,

(ii) All slacks are zero.

3.2. Estimating returns to scale of efficient DMUs in the presence of undesirable inputs and outputs

The dual (multiplier) form associated with model (7) is as follows:

$$
\begin{aligned}
Min \quad & \sum_{i=1}^{m_2} v_i^b x_{ip}^b - \sum_{r=1}^{s_2} u_r^b y_{rp}^b + u_p + t_p - w_p \\
s.t. \quad & \sum_{i=1}^{m_1} v_i^g x_{ij}^g + \sum_{i=1}^{m_2} v_i^b x_{ij}^b - \sum_{r=1}^{s_1} u_r^g y_{rj}^g - \sum_{r=1}^{s_2} u_r^b y_{rj}^b + u_p \geq 0, \quad j = 1,\dots,n, \\
& \sum_{i=1}^{m_1} v_i^g x_{ip}^g - t_p = 1, \qquad (8) \\
& \sum_{r=1}^{s_1} u_r^g y_{rp}^g - w_p = 1, \\
& v_i^g \geq 0, \quad i = 1,\dots,m_1, \\
& v_i^b \geq 0, \quad i = 1,\dots,m_2, \\
& u_r^g \geq 0, \quad i = 1,\dots,s_1, \\
& u_r^b \geq 0, \quad i = 1,\dots,s_2.
\end{aligned}
$$

By considering variable RTS assumption, we have the following production possibility set (PPS):

$$
PPS = \left\{ \left(\mathbf{X}^g, \mathbf{X}^b, \mathbf{Y}^g, \mathbf{Y}^b \right) \middle| \sum_{j=1}^{n} \lambda_j \mathbf{X}_j^g \leq \mathbf{X}^g, \sum_{j=1}^{n} \lambda_j \mathbf{X}_j^b \leq \mathbf{X}^b, \sum_{j=1}^{n} \lambda_j \mathbf{Y}_j^g \geq \mathbf{Y}^g, \sum_{j=1}^{n} \lambda_j \mathbf{Y}_j^b \geq \mathbf{Y}^b, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0; j = 1,2,\dots,n \right\}. \qquad (9)
$$

Let

$$
\alpha(\beta) = \max \left\{ \alpha \middle| \left(\beta \mathbf{X}^g, \beta \mathbf{X}^b, \alpha \mathbf{Y}^g, \alpha \mathbf{Y}^b \right) \in PPS \right\}, \qquad (10)
$$

then, we define γ^+ and γ^- as bellows:

$$
\gamma^+ = \lim_{\beta \to 1^+} \frac{\alpha(\beta) - 1}{\beta - 1}, \qquad \gamma^- = \lim_{\beta \to 1^-} \frac{\alpha(\beta) - 1}{\beta - 1}.
$$

Theorem 4. *Suppose that* DMU_p *with input-output combination* $\left(\mathbf{X}_p^g, \mathbf{X}_p^b, \mathbf{Y}_p^g, \mathbf{Y}_p^b \right)$ *is efficient DMU by using model (7). Then, we have:*

(i) $\gamma^+ > 1$ *and* $\gamma^- > 1$ *if and only if* DMU_p *has increasing RTS (IRS).*

(ii) $\gamma^+ < 1$ *and* $\gamma^- < 1$ *if and only if* DMU_p *has decreasing RTS (DRS).*

(iii) $\gamma^+ < 1$ *and* $\gamma^- > 1$ *if and only if* DMU_p *has constant RTS (CRS).*

Proof. *Case (i):* First, suppose that DMU_p has IRS and also, assume that $\mathbf{Y} = \left(\mathbf{Y}^g, \mathbf{Y}^b\right) = F\left(\mathbf{X}^g, \mathbf{X}^b\right) = F\left(\mathbf{X}\right)$ is production function. Therefore according to the definition of increasing RTS, we have:

$$\begin{cases} \forall \beta : \beta > 1 \Rightarrow F\left(\beta \mathbf{X}_p^g, \beta \mathbf{X}_p^b\right) > \beta F\left(\mathbf{X}_p^g, \mathbf{X}_p^b\right), & (11) \\ \& \\ \forall \beta : \beta < 1 \Rightarrow F\left(\beta \mathbf{X}_p^g, \beta \mathbf{X}_p^b\right) < \beta F\left(\mathbf{X}_p^g, \mathbf{X}_p^b\right). & (12) \end{cases}$$

Thus, associated with (10), (11), and (12):

$$\begin{cases} \left(\alpha(\beta)\mathbf{Y}_p^g, \alpha(\beta)\mathbf{Y}_p^b\right) > \beta\left(\mathbf{Y}_p^g, \mathbf{Y}_p^b\right) \Rightarrow \alpha(\beta)\mathbf{Y}_p > \beta \mathbf{Y}_p \Rightarrow \alpha(\beta) y_{rp} > \beta y_{rp}, \quad (r = 1, \dots, s), \\ \& \\ \left(\alpha(\beta)\mathbf{Y}_p^g, \alpha(\beta)\mathbf{Y}_p^b\right) < \beta\left(\mathbf{Y}_p^g, \mathbf{Y}_p^b\right) \Rightarrow \alpha(\beta)\mathbf{Y}_p < \beta \mathbf{Y}_p \Rightarrow \alpha(\beta) y_{rp} < \beta y_{rp}, \quad (r = 1, \dots, s), \end{cases}$$

$$\Rightarrow \begin{cases} \alpha(\beta) > \beta \Rightarrow \alpha(\beta) - 1 > \beta - 1, \\ \& \\ \alpha(\beta) < \beta \Rightarrow \alpha(\beta) - 1 < \beta - 1, \end{cases}$$

$$\Rightarrow \begin{cases} \gamma^+ > 1, \\ \& \\ \gamma^- > 1. \end{cases}$$

Conversely, suppose that $\gamma^+ > 1$ and $\gamma^- > 1$ then, we respectively have:

$$\begin{cases} \dfrac{\alpha(\beta)-1}{\beta-1} > 1 \Rightarrow \alpha(\beta)-1 > \beta-1, \\ \& \\ \dfrac{\alpha(\beta)-1}{\beta-1} > 1 \Rightarrow \alpha(\beta)-1 < \beta-1, \end{cases}$$

$$\Rightarrow \begin{cases} \alpha(\beta) > \beta \Rightarrow \alpha(\beta) y_{rp} > \beta y_{rp}, \quad (r=1,\ldots,s) \Rightarrow \alpha(\beta)\mathbf{Y}_p > \beta\mathbf{Y}_p, \\ \& \\ \alpha(\beta) < \beta \Rightarrow \alpha(\beta) y_{rp} < \beta y_{rp}, \quad (r=1,\ldots,s) \Rightarrow \alpha(\beta)\mathbf{Y}_p < \beta\mathbf{Y}_p, \end{cases}$$

$$\Rightarrow \begin{cases} F\left(\beta\mathbf{X}_p\right) = F\left(\beta\mathbf{X}_p^g, \beta\mathbf{X}_p^b\right) > \beta F\left(\mathbf{X}_p\right) = \beta F\left(\mathbf{X}_p^g, \mathbf{X}_p^b\right), \\ \& \\ F\left(\beta\mathbf{X}_p\right) = F\left(\beta\mathbf{X}_p^g, \beta\mathbf{X}_p^b\right) < \beta F\left(\mathbf{X}_p\right) = \beta F\left(\mathbf{X}_p^g, \mathbf{X}_p^b\right). \end{cases}$$

Therefore according to the definition of increasing RTS, DMU_p has IRS.
Other cases can be proved, similarly. □

Suppose that $\left(\mathbf{V}^{g^*}, \mathbf{V}^{b^*}, \mathbf{U}^{g^*}, \mathbf{U}^{b^*}, u_p^*, t_p^*, w_p^*\right)$ is an obtained optimal solution from model (8).

Theorem 5. *Suppose that DMU_p with input-output combination $\left(\mathbf{X}_p^g, \mathbf{X}_p^b, \mathbf{Y}_p^g, \mathbf{Y}_p^b\right)$ is efficient DMU by using model (7). Then, we have:*

(i) DMU_p has IRS if and only if $u_p^* < 0$ for all optimal solutions of model (8).

(ii) DMU_p has DRS if and only if $u_p^* > 0$ for all optimal solutions of model (8).

(iii) DMU_p has CRS if and only if $u_p^* = 0$ for at least one optimal solution of model (8).

Proof. *Case (i):* First, suppose DMU_p has IRS. Then according to Theorem 4, $\gamma^+ > 1$ and $\gamma^- > 1$. Since $\gamma^+ > 1$, we include $\alpha(\beta) > \beta$. Moreover, DMU_p is efficient, therefore:

$$\left(\mathbf{V}^{g^*}\right)^T \mathbf{X}_p^g + \left(\mathbf{V}^{b^*}\right)^T \mathbf{X}_p^b - \left(\mathbf{U}^{g^*}\right)^T \mathbf{Y}_p^g - \left(\mathbf{U}^{b^*}\right)^T \mathbf{Y}_p^b + u_p^* = 0. \tag{13}$$

According to (10), we imply that:

$$\left(\mathbf{V}^{g^*}\right)^T \left(\beta\mathbf{X}_p^g\right) + \left(\mathbf{V}^{b^*}\right)^T \left(\beta\mathbf{X}_p^b\right) - \left(\mathbf{U}^{g^*}\right)^T \left(\alpha(\beta)\mathbf{Y}_p^g\right) - \left(\mathbf{U}^{b^*}\right)^T \left(\alpha(\beta)\mathbf{Y}_p^b\right) + u_p^* = 0. \tag{14}$$

Since $\alpha(\beta) > \beta$, so associated with (14), we have:

$$\beta\left(\left(\mathbf{V}^{g^*}\right)^T \left(\mathbf{X}_p^g\right)\right) + \beta\left(\left(\mathbf{V}^{b^*}\right)^T \left(\mathbf{X}_p^b\right)\right) - \beta\left(\left(\mathbf{U}^{g^*}\right)^T \left(\mathbf{Y}_p^g\right)\right) - \beta\left(\left(\mathbf{U}^{b^*}\right)^T \left(\mathbf{Y}_p^b\right)\right) + u_p^* > 0,$$

$$\Rightarrow \beta\left(\left(\mathbf{V}^{g^*}\right)^T \left(\mathbf{X}_p^g\right) + \left(\mathbf{V}^{b^*}\right)^T \left(\mathbf{X}_p^b\right) - \left(\mathbf{U}^{g^*}\right)^T \left(\mathbf{Y}_p^g\right) - \left(\mathbf{U}^{b^*}\right)^T \left(\mathbf{Y}_p^b\right) + u_p^*\right) + u_p^*(1-\beta) > 0.$$

Thus, according to (13), we include that $u_p^*(1-\beta) > 0$. Since $\beta > 1$ then $u_p^* < 0$.

Similarly, for $\gamma^- > 1$, we obtain $u_p^*(1-\beta) < 0$ and imply $u_p^* < 0$.

Conversely, assume that $u_p^* < 0$ for all optimal solutions of model (8). Now consider $\mathbf{Z}_\delta$ as below:

$$\mathbf{Z}_\delta = \left((1+\delta)\mathbf{X}_p^g, (1+\delta)\mathbf{X}_p^b, (1+\delta)\mathbf{Y}_p^g, (1+\delta)\mathbf{Y}_p^b\right),$$

where δ is a small positive number.
Therefore,

$$\begin{aligned}&\left(\mathbf{V}^{g^*}\right)^T \left((1+\delta)\mathbf{X}_p^g\right) + \left(\mathbf{V}^{b^*}\right)^T \left((1+\delta)\mathbf{X}_p^b\right) - \left(\mathbf{U}^{g^*}\right)^T \left((1+\delta)\mathbf{Y}_p^g\right) - \left(\mathbf{U}^{b^*}\right)^T \left((1+\delta)\mathbf{Y}_p^b\right) + u_p^*,\\ &= (1+\delta)\left(\left(\mathbf{V}^{g^*}\right)^T \mathbf{X}_p^g + \left(\mathbf{V}^{b^*}\right)^T \mathbf{X}_p^b - \left(\mathbf{U}^{g^*}\right)^T \mathbf{Y}_p^g - \left(\mathbf{U}^{b^*}\right)^T \mathbf{Y}_p^b + u_p^*\right) - \delta u_p^*.\end{aligned} \tag{15}$$

Thus according to (13) and (15), we include that $-\delta u_p^* > 0$. So, $\mathbf{Z}_\delta$ does not lie on the efficient frontier. Hence, DMU_p has IRS.

Other cases can be proved, similarly. □

Consider the following non-radial DEA model for evaluating DMU_p in the presence of undesirable factors:

$$
\begin{aligned}
Max \quad & \sum_{i=1}^{m_1} s_i^{-g} + \sum_{r=1}^{s_1} s_r^{+g} - \sum_{i=1}^{m_2} s_i^{-b} - \sum_{r=1}^{s_2} s_r^{+b} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^{g} + s_i^{-g} = x_{ip}^{g}, \quad i = 1,\dots,m_1, \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^{b} + s_i^{-b} = x_{ip}^{b}, \quad i = 1,\dots,m_2, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{g} - s_r^{+g} = y_{rp}^{g}, \quad r = 1,\dots,s_1, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^{b} - s_r^{+b} = y_{rp}^{b}, \quad r = 1,\dots,s_2, \qquad\qquad (16) \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0, \quad j = 1,\dots,n, \\
& s_i^{-g} \geq 0, \quad i = 1,\dots,m_1, \qquad s_r^{+g} \geq 0, \quad j = 1,\dots,s_1, \\
& s_i^{-b} \geq 0, \quad i = 1,\dots,m_2, \qquad s_r^{+b} \geq 0, \quad j = 1,\dots,s_2.
\end{aligned}
$$

Definition 4. DMU_p is called efficient under model (16) if and only if the optimal value of its objective function is zero.

Theorem 6. *Suppose that* DMU_p *with input-output combination* $\left(\mathbf{X}_p^g, \mathbf{X}_p^b, \mathbf{Y}_p^g, \mathbf{Y}_p^b\right)$ *is efficient DMU by using model (7). Then, we have*

(i) *There is* $\xi > 1$ *so that* $\left(\xi\mathbf{X}_p^g, \xi\mathbf{X}_p^b, \xi\mathbf{Y}_p^g, \xi\mathbf{Y}_p^b\right) \in PPS$ *is inefficient if and only if* DMU_p *has IRS.*

(ii) *There is* $0 < \xi < 1$ *so that* $\left(\xi\mathbf{X}_p^g, \xi\mathbf{X}_p^b, \xi\mathbf{Y}_p^g, \xi\mathbf{Y}_p^b\right) \in PPS$ *is inefficient if and only if* DMU_p *has DRS.*

(iii) *There is* $\xi > 0$ *so that* $\left(\xi\mathbf{X}_p^g, \xi\mathbf{X}_p^b, \xi\mathbf{Y}_p^g, \xi\mathbf{Y}_p^b\right) \in PPS$ *is efficient if and only if* DMU_p *has CRS.*

Proof. *Case (i):* Assume that $\left(\mathbf{V}^{g^*}, \mathbf{V}^{b^*}, \mathbf{U}^{g^*}, \mathbf{U}^{b^*}, u_p^*, t_p^*, w_p^*\right)$ is an obtained optimal solution from model (8) in assessing DMU_p. Since DMU_p is efficient, so:

$$\left(\mathbf{V}^{g^*}\right)^T \mathbf{X}_p^g + \left(\mathbf{V}^{b^*}\right)^T \mathbf{X}_p^b - \left(\mathbf{U}^{g^*}\right)^T \mathbf{Y}_p^g - \left(\mathbf{U}^{b^*}\right)^T \mathbf{Y}_p^b + u_p^* = 0. \qquad (17)$$

Also, $\left(\xi\mathbf{X}_p^g,\xi\mathbf{X}_p^b,\xi\mathbf{Y}_p^g,\xi\mathbf{Y}_p^b\right)\in PPS$ is inefficient, thus we have:

$$\left(\mathbf{V}^{g^*}\right)^T\left(\xi\mathbf{X}_p^g\right)+\left(\mathbf{V}^{b^*}\right)^T\left(\xi\mathbf{X}_p^b\right)-\left(\mathbf{U}^{g^*}\right)^T\left(\xi\mathbf{Y}_p^g\right)-\left(\mathbf{U}^{b^*}\right)^T\left(\xi\mathbf{Y}_p^b\right)+u_p^*>0,$$

$$\Rightarrow\xi\left(\left(\mathbf{V}^{g^*}\right)^T\left(\mathbf{X}_p^g\right)+\left(\mathbf{V}^{b^*}\right)^T\left(\mathbf{X}_p^b\right)-\left(U^{g^*}\right)^T\left(\mathbf{Y}_p^g\right)-\left(\mathbf{U}^{b^*}\right)^T\left(\mathbf{Y}_p^b\right)+u_p^*\right)+u_p^*-\xi u_p^*>0. \tag{18}$$

Therefore according to (17) and (18), we conclude that $u_p^*(1-\xi)>0$. Since $\xi>1$ then $u_p^*<0$. Thus associated with Theorem 5, DMU_p has IRS.

Conversely, suppose that DMU_p has IRS. Then according to Theorem 5, $u_p^*<0$. Contrary: assume that for each $\xi>1$, $\left(\xi\mathbf{X}_p^g,\xi\mathbf{X}_p^b,\xi\mathbf{Y}_p^g,\xi\mathbf{Y}_p^b\right)\in PPS$ is efficient. Therefore, each convex combination of $\left(\mathbf{X}_p^g,\mathbf{X}_p^b,\mathbf{Y}_p^g,\mathbf{Y}_p^b\right)$ and $\left(\xi\mathbf{X}_p^g,\xi\mathbf{X}_p^b,\xi\mathbf{Y}_p^g,\xi\mathbf{Y}_p^b\right)$ lies on the efficient frontier. Thus, there is supporting hyperplane $\left(\bar{\mathbf{V}}^g\right)^T\mathbf{X}^g+\left(\bar{\mathbf{V}}^b\right)^T\mathbf{X}^b-\left(\bar{\mathbf{U}}^g\right)^T\mathbf{Y}^g-\left(\bar{\mathbf{U}}^b\right)^T\mathbf{Y}^b+\bar{u}_p=0$ of PPS which it passes from $\left(\mathbf{X}_p^g,\mathbf{X}_p^b,\mathbf{Y}_p^g,\mathbf{Y}_p^b\right)$ and $\left(\xi\mathbf{X}_p^g,\xi\mathbf{X}_p^b,\xi\mathbf{Y}_p^g,\xi\mathbf{Y}_p^b\right)$. So, if $\left(\bar{\mathbf{V}}^g\right)^T\mathbf{X}_p^g-\bar{t}_p=\alpha$ and $\left(\bar{\mathbf{U}}^g\right)^T\mathbf{Y}_p^g-\bar{w}_p=\beta$ then, the following optimal solution of model (8) in assessing DMU_p which is active on $\left(\mathbf{X}_p^g,\mathbf{X}_p^b,\mathbf{Y}_p^g,\mathbf{Y}_p^b\right)$ and $\left(\xi\mathbf{X}_p^g,\xi\mathbf{X}_p^b,\xi\mathbf{Y}_p^g,\xi\mathbf{Y}_p^b\right)$:

$$\begin{aligned}&\left(\mathbf{V}^{g^*},\mathbf{V}^{b^*},\mathbf{U}^{g^*},\mathbf{U}^{b^*},u_p^*,t_p^*,w_p^*\right)\\&=\Bigg(\alpha^{-1}\bar{\mathbf{V}}^g,\left(\left(\bar{\mathbf{U}}^b\right)^T\mathbf{Y}_p^b-\left(\bar{\mathbf{V}}^b\right)^T\mathbf{X}_p^b\right)^{-1}\bar{\mathbf{V}}^b,\beta^{-1}\bar{\mathbf{U}}^g,\left(\left(\bar{\mathbf{U}}^b\right)^T\mathbf{Y}_p^b-\left(\bar{\mathbf{V}}^b\right)^T\mathbf{X}_p^b\right)^{-1}\bar{\mathbf{U}}^b,1-\alpha^{-1}\bar{t}_p+\beta^{-1}\bar{w}_p,\\&\qquad\alpha^{-1}\bar{t}_p,\beta^{-1}\bar{w}_p\Bigg).\end{aligned}$$

Hence, we have:

$$\left(V^{g^*}\right)^T\left(X_p^g\right)+\left(V^{b^*}\right)^T\left(X_p^b\right)-\left(U^{g^*}\right)^T\left(Y_p^g\right)-\left(U^{b^*}\right)^T\left(Y_p^b\right)+u_p^*=0, \tag{19}$$

$$\left(\mathbf{V}^{g^*}\right)^T\left(\xi\mathbf{X}_p^g\right)+\left(\mathbf{V}^{b^*}\right)^T\left(\xi\mathbf{X}_p^b\right)-\left(\mathbf{U}^{g^*}\right)^T\left(\xi\mathbf{Y}_p^g\right)-\left(\mathbf{U}^{b^*}\right)^T\left(\xi\mathbf{Y}_p^b\right)+u_p^*=0. \tag{20}$$

Thus, according to (19) and (20):

$$\xi\left(\left(\mathbf{V}^{g^*}\right)^T\left(\mathbf{X}_p^g\right)+\left(\mathbf{V}^{b^*}\right)^T\left(\mathbf{X}_p^b\right)-\left(\mathbf{U}^{g^*}\right)^T\left(\mathbf{Y}_p^g\right)-\left(\mathbf{U}^{b^*}\right)^T\left(\mathbf{Y}_p^b\right)+u_p^*\right)+u_p^*-\xi u_p^*=0. \tag{21}$$

So, associated with (17) and (21), we imply that $u_p^*(1-\xi)=0$. Since $\xi>1$ then $u_p^*=0$. Hence, according to Theorem 5, DMU_p has CRS and it is a contradiction. Thus, the contrary suppose is false and the proof is complete.

Other cases can be proved, similarly. □

Now in order to estimate returns to scale of DMU_p, we present the following non-radial DEA model:

$$
\begin{aligned}
Max \quad & \Gamma^p = \sum_{i=1}^{m_1} s_i^{-g} + \sum_{r=1}^{s_1} s_r^{+g} - \sum_{i=1}^{m_2} s_i^{-b} - \sum_{r=1}^{s_2} s_r^{+b} \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij}^g + s_i^{-g} = \xi x_{ip}^g, \quad i=1,\dots,m_1, \\
& \sum_{j=1}^{n} \lambda_j x_{ij}^b + s_i^{-b} = \xi x_{ip}^b, \quad i=1,\dots,m_2, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^g - s_r^{+g} = \xi y_{rp}^g, \quad r=1,\dots,s_1, \\
& \sum_{j=1}^{n} \lambda_j y_{rj}^b - s_r^{+b} = \xi y_{rp}^b, \quad r=1,\dots,s_2, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \ge 0, \quad j=1,\dots,n, \\
& s_i^{-g} \ge 0, \quad i=1,\dots,m_1, \qquad s_r^{+g} \ge 0, \quad j=1,\dots,s_1, \\
& s_i^{-b} \ge 0, \quad i=1,\dots,m_2, \qquad s_r^{+b} \ge 0, \quad j=1,\dots,s_2.
\end{aligned}
\tag{22}
$$

Let $\left(\Gamma^{p^*}, \boldsymbol{\lambda}^*, \xi^*, \mathbf{S}^{-g^*}, \mathbf{S}^{-b^*}, \mathbf{S}^{+g^*}, \mathbf{S}^{+b^*}\right)$ is an obtained optimal solution of model (22).

The following theorem is provided to identify returns to scale of DMU_p by using model (22).

Theorem 7. *Suppose that DMU_p with input-output combination $\left(\mathbf{X}_p^g, \mathbf{X}_p^b, \mathbf{Y}_p^g, \mathbf{Y}_p^b\right)$ is efficient by using model (7). The following conditions estimate returns to scale of evaluated DMU_p by model (22):*

(i) The optimal value of the objective function is non-zero and $\xi^*>1$ if and only if DMU_p has IRS.

(ii) The optimal value of the objective function is non-zero and $0<\xi^*<1$ if and only if DMU_p has DRS.

(iii) The optimal value of the objective function is zero if and only if DMU_p has CRS.

Proof. *Case (i):* Assume that the optimal value of the objective function of model (22) is non-zero and $\xi^* > 1$. Therefore, $\left(\xi^*\mathbf{X}_p^g, \xi^*\mathbf{X}_p^b, \xi^*\mathbf{Y}_p^g, \xi^*\mathbf{Y}_p^b\right) \in PPS$ is inefficient under model (16). So, associated with Theorem 6, DMU_p has IRS.

Conversely, let DMU_p has IRS. Then according to Theorem 6, there is $\xi > 1$ so that $\left(\xi^*\mathbf{X}_p^g, \xi^*\mathbf{X}_p^b, \xi^*\mathbf{Y}_p^g, \xi^*\mathbf{Y}_p^b\right) \in PPS$ is inefficient. Thus, inefficiency of $\left(\xi^*\mathbf{X}_p^g, \xi^*\mathbf{X}_p^b, \xi^*\mathbf{Y}_p^g, \xi^*\mathbf{Y}_p^b\right)$ implies that the value of its objective function is non-zero. Since model (16) is as maximization, so in evaluating $\left(\xi^*\mathbf{X}_p^g, \xi^*\mathbf{X}_p^b, \xi^*\mathbf{Y}_p^g, \xi^*\mathbf{Y}_p^b\right)$, the optimal value of its objective function must be non-zero. Hence, $\left(\xi^*\mathbf{X}_p^g, \xi^*\mathbf{X}_p^b, \xi^*\mathbf{Y}_p^g, \xi^*\mathbf{Y}_p^b\right) \in PPS$ is inefficient. Now, we must prove that $\xi^* > 1$. Contrary: suppose that $\xi^* \le 1$. If $\xi^* < 1$ than according to Theorem 6, DMU_p has DRS and also, if $\xi^* = 1$ then DMU_p is inefficient. Thus, there are two contradictions. Hence, the contrary suppose is false and the proof is complete.

Other cases can be proved, similarly. $\square$

In the next section, we explicitly survey an empirical example to highlight the proposed RTS approach.

4. Empirical example

In this section, we apply our proposed RTS approach on 20 Greek schools to estimate returns to scale which their set of inputs and outputs has been shown in Table 1. These schools have 3 inputs as: budget, facilities index (desirable inputs), and stupid students (undesirable input). Furthermore, they have 4 outputs as: excellent graduated students, admission (desirable outputs), lazy graduated students, and expelled students (undesirable outputs). Data of inputs and outputs of schools has been listed in Table 2.

Table 1.
The set of inputs and outputs of schools .

Desirable and undesirable inputs		Desirable and undesirable Outputs	
(I_1)	Budget (D)	(O_1)	Excellent graduated students (D)
(I_2)	Facilities index (D)	(O_2)	Admission (D)
(I_3)	Stupid students (U)	(O_3)	Lazy graduated students (U)
		(O_4)	Expelled students (U)

Table 2.
Data of inputs and outputs of schools.

School	I_1	I_2	I_3	O_1	O_2	O_3	O_4
S1	23940	6	11	19	10	9	5
S2	25450	5	9	38	14	11	2
S3	24000	4	5	34	4	7	1
S4	26500	7	18	29	4	10	8
S5	31200	6	4	48	11	9	9
S6	32600	5	3	36	17	8	4
S7	31580	5	17	73	18	6	4
S8	35600	5	6	40	22	13	3
S9	39160	4	23	33	38	12	1
S10	42800	4	8	62	13	4	8
S11	42840	7	12	78	27	14	4
S12	41000	4	10	62	27	5	9
S13	45980	7	4	70	28	5	5
S14	51000	7	3	59	15	10	2
S15	52200	5	11	76	25	16	7
S16	56000	7	19	56	26	9	3
S17	56700	7	7	59	33	15	4
S18	58140	4	21	78	34	2	6
S19	52000	4	6	96	18	7	6
S20	60100	7	6	95	35	3	5

Table 3.
The obtained results from model (7) and definition 3.

School	θ_p^*	φ_p^*	z_p^*	$s_1^{-g^*}$	$s_2^{-g^*}$	$s_1^{+g^*}$	$s_2^{+g^*}$	Results of definition 3
S1	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Efficient
S2	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Efficient
S3	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Efficient
S4	1.0000	1.6160	0.6160	0.0000	2.6702	0.0000	2.1536	Inefficient
S5	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Efficient
S7	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Efficient
S8	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Efficient
S9	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Efficient
S10	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Efficient
S11	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Efficient
S12	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Efficient
S13	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Efficient
S14	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Efficient
S15	0.8752	1.0000	0.1248	0.0000	0.0000	0.0000	0.0000	Inefficient

S16	0.6953	1.0858	0.3905	0.0000	0.0000	0.0000	0.0000	Inefficient
S17	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Efficient
S18	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Efficient
S19	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Efficient
S20	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	Efficient

As can be seen in Table 3, by using the proposed method, schools 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 17, 18, 19, and 20 are efficient and schools 4, 15, and 16 are inefficient. We are going to estimate returns to scale of these 17 efficient schools by our proposed RTS approach. Table 4 represents the obtained results from the proposed RTS approach.

Table 4.
The obtained results from model (22).

Efficient school	ξ^*	$s_1^{-g^*}$	$s_2^{-g^*}$	$s_1^{-b^*}$	$s_1^{+g^*}$	$s_2^{+g^*}$	$s_1^{+b^*}$	$s_2^{+b^*}$	Γ^{p^*}	Results of the proposed approach
S1	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	CRS [1]
S2	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	CRS
S3	1.8775	5.6E+3	1.1887	0.0000	0.0000	1.6E+1	0.0000	2.4167	5.6E+3	IRS [2]
S5	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	CRS
S6	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	CRS
S7	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	CRS
S8	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	CRS
S9	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	CRS
S10	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	CRS
S11	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	CRS
S12	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	CRS
S13	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	CRS
S14	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	CRS
S17	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	CRS
S18	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	CRS
S19	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	CRS
S20	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	CRS

[1] Constant returns to scale.
[2] Increasing returns to scale.

As presented in Table 4, RTS of efficient schools can be estimated by Γ^{p^*} and ξ^*. According to Table 4 and Theorem 7, S3 has IRS because $\Gamma^{p^*} = 5.6\text{E+}3 > 0$ and $\xi^* = 1.8775 > 1$ while other efficient schools have CRS because $\Gamma^{p^*} = 0$ and $\xi^* = 1.0000 > 0$.

5. Conclusion and future extensions

In this research, we first introduce a new input-output oriented model for determining efficient DMUs in the presence of undesirable (bad) inputs and outputs factors and then, a new non-radial model is presented to estimate RTS of these DMUs in DEA. Thus, this paper opens up the new RTS approach which determines RTS in the presence of undesirable factors. In this vein, using an illustrative empirical example is also summarized in Tables.

In the DEA literature, there are many RTS approaches for estimating RTS of DMUs with desirable (good) data while in the real world, both desirable (good) and undesirable (bad) inputs and outputs may be present. Our proposed RTS approach is capable of identifying RTS of efficient DMUs in the presence of undesirable factors which is a advantage of this study.

Note that, since an inefficient DMU has more than one projection on the empirical function so, different returns to scales can be obtained for projections of the inefficient DMU by using the proposed RTS approach.

It is necessary to mention that, this article can be similarly extended for special desirable and undesirable data such as; interval, integer, stochastic, fuzzy, and etc.

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