



A Non-radial rough DEA model

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Abstract

For efficiency evaluation of some of the Decision Making Units that have uncertain information, Rough Data Envelopment Analysis technique is used, which is derived from rough set theorem and Data Envelopment Analysis (DEA). In some situations rough data alter nonradially. To this end, this paper proposes additive rough-DEA model and illustrates the proposed model by a numerical example.

Keywords: Rough data envelopment analysis (RDEA), Data envelopment analysis (DEA), Uncertainty

1 Introduction

Data envelopment analysis (DEA), was first put forward by Charnes, Cooper and Rhodes (CCR model) in 1978 [1], it is performance measurement technique which can be used for evaluating the relative efficiency of Decision Making Units (DMU's) in organizations.

One of research of DEA is Rough-DEA that researches on Rough-DEA (RDEA) are still very restricted. Therefore, the research on combining DEA with Rough set theory is an attractive study field.

Pawlak introduced a theory of Rough sets in 1982. Since then rough-set theory has been developed very rapidly and has resulted in a number of applications.

In order to provide an axiomatic theory to describe rough events, Liu [2], established the thrust theory which is branch of mathematics that studies the behavior of rough events.

Liu [2], defined a rough variable ξ by a measurable function from a rough space $(\mu, \Delta, \Lambda, \pi)$ to the set of real numbers, that is for any Borel set B of \mathbb{R} , $\{\lambda; \xi(\lambda) \in B\} \in \Lambda$ hold – Based on the thrust theory, Liu and Jiuping Xu, bin Liu, Desheng Wu [2,3], studied some rough programming models with variables as parameters. The remainder of this paper is organized as follows.

In section 2 we want to bring some concepts about measurable space, Rough space and trust measure.

In section 3 we present the Additive Rough DEA (ARDEA) model, section 4 provides an application example to illustrate the efficiency of the ARDEA model. Finally, concluding remarks are outlined in section 5.

2 Background

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This section recalls several concepts of rough variable theory to characterize the rough uncertainty.

Definition 2.1[2]: Let μ be a nonempty sets, a collection Λ is called a σ – algebra of subsets of μ if is satisfied in the following conditions:

(a) $\mu \in \Lambda$

(b) If $A \in \Lambda$, then $A^c \in \Lambda$

(c) If $A_i \in \Lambda$, for $i = 1 \dots$ then $\bigcup_{i=1}^{\infty} A_i \in A$

Definition 2.2[2]: Let μ be a nonempty set, and Λ a σ – algebra of subsets of μ , then (μ, Λ) is called a measurable space and the elements in Λ are called measurable sets furthermore, if π is measure defined on (μ, Λ) , then triplet (μ, Λ, π) is called measure space.

Definition 2.3[2]: Let μ be a nonempty set, Λ a σ – algebra of subsets of μ , Δ an element in Λ , and π a set function on Λ , then $(\mu, \Delta, \Lambda, \pi)$ is called rough space, furthermore, the triplet (μ, Λ, π) is a measure space.

Definition 2.4[2]: Uncertain variables: An uncertain variable is a measurable function from an uncertainty space (μ, Λ, π) to the set of real numbers, i.e., for any Borel set B of real numbers, the set $\{\xi \in B\} = \{\gamma \in \mu(\xi(\gamma) \in B)\}$ is an event.

Example [1]: Random variables, fuzzy variables and hybrid variable are instance of uncertain variable.

Definition 2.5[2]: Uncertain vector: An n-dimensional uncertain vector is a measurable function from an uncertainty space (μ, A, π) to the set of n-dimensional real vectors, i.e., for any Borel set B of \mathbb{R}^n , the $\{\xi \in B\} = \{\gamma \in \mu \mid \xi(\gamma) \in B\}$ is an event.

Definition 2.6[2]: Let (μ, Δ, A, π) be a rough space then the upper trust of event $A \in A$ is defined by

$$T_r^+ \{A\} = \frac{\pi\{A\}}{\pi\{\mu\}}.$$

The lower trust of the event A is defined by $T_r^- \{A\} = \frac{\pi\{A \cap \Delta\}}{\pi\{\Delta\}}$ and the trust of the event A is defined

$$\text{by } T_r \{A\} = \frac{1}{2} (T_r^+ \{A\} + T_r^- \{A\}).$$

Definition 2.7[2]: Let ξ be a rough variable, and $\alpha \in (0, 1]$, then $\xi_{\text{sup}}(\alpha) = \sup\{r \mid Tr(\xi \geq r) \geq \alpha\}$ is called the α – optimistic value to ξ and $\xi_{\text{inf}}(\alpha) = \inf\{r \mid Tr(\xi \leq r) \geq \alpha\}$ is called the α – pessimistic value to ξ .

It is important to have focus that $\xi = ([a, b], [c, d])$ be a rough variable with $c \leq a \leq b \leq d$ then the α – optimistic value of ξ is

$$\xi_{\text{sup}}(\alpha) = \begin{cases} (1 - 2\alpha)d + 2\alpha c & \text{if } \alpha \geq ((d - b)/2[d - c]) \\ 2(1 - \alpha)d + (2\alpha - 1)c & \text{if } \alpha \geq ((2d - a - c)/[2(d - c)]) \\ \frac{d(b - a) + b(d - c) - 2\alpha(b - a)(d - c)}{(b - a) + (d - c)} & \text{otherwise} \end{cases}$$

And the α – pessimistic value of ξ is

$$\xi_{\inf}(\alpha) = \begin{cases} (1-2\alpha)c + 2\alpha d & \text{if } \alpha \leq \frac{a-c}{2(d-c)} \\ 2(1-\alpha)c + (2\alpha-1)d & \text{if } \alpha \geq \frac{b+d-2c}{2(d-c)} \\ \frac{c(b-a) + a(d-c) + 2\alpha(b-a)(d-c)}{(b-a) + (d-c)} & \text{otherwise} \end{cases}$$

Theorem 1. [2]. Let $\xi_{\inf}(\alpha)$ and $\xi_{\sup}(\alpha)$ be the α -pessimistic and α -optimistic values of the rough variable ξ , respectively – then we have

- (a) $Tr\{\xi \geq \xi_{\sup}(\alpha)\} \geq \alpha$ And $Tr\{\xi \leq \xi_{\inf}(\alpha)\} \geq \alpha$;
- (b) $\xi_{\inf}(\alpha)$ is an increasing and left-continuous function of α ;
- (c) $\xi_{\sup}(\alpha)$ is a decreasing and left-continuous function of α ;
- (d) If $0 < \alpha \leq 1$, then $\xi_{\inf}(\alpha) = \xi_{\sup}(\alpha)(1-\alpha)$, and $\xi_{\sup}(\alpha) = \xi_{\inf}(1-\alpha)$
- (e) If $0 < \alpha \leq 0.5$, then $\xi_{\inf}(\alpha) \leq \xi_{\sup}(\alpha)$;
- (f) If $0.5 < \alpha \leq 1$, then $\xi_{\inf}(\alpha) \leq \xi_{\sup}(\alpha)$;

Definition 2.7[3]: Dmu_o is rough DEA efficient if its best possible upper bound efficiency $(\theta^*)^{\inf(\alpha)}$ is equal to One; otherwise, it is said to be rough DEA inefficient if $(\theta^*)^{\inf(\alpha)} < 1$.

3 Additive rough DEA

Let there are n DMUs and each DMU has m inputs and s outputs, the input and output vectors of DMU_j are rough vectors i.e.

$$\hat{x}_j = \left(\hat{x}_{ij}, \dots, \hat{x}_{mj} \right) > 0 \quad \text{and} \quad \hat{y}_j = \left(\hat{y}_{ij}, \dots, \hat{y}_{sj} \right) > 0 \quad j = 1, \dots, n$$

According to the Additive DEA model (the following model), we can formulate a DEA model with rough.

$$\begin{aligned} & \text{Max} \left(\sum_{i=1}^m S_i^- + \sum_{r=1}^s S_r^+ \right) \\ & \text{s.t.} \quad \sum_{j=1}^n \lambda_j x_{ij} + S_i^- = x_{io} \quad i = 1, \dots, m \\ & \quad \quad \sum_{j=1}^n \lambda_j y_{rj} - S_r^+ = y_{ro} \quad r = 1, \dots, s \\ & \quad \quad \lambda_j \geq 0. \quad j = 1, \dots, n \\ & \quad \quad S^- \geq, S^+ \geq 0 \end{aligned} \tag{1}$$

Generally, to deal with the rough uncertainty discussed above, Liu [2], proposed the rough expected value model (EVM) and α – optimistic value and α – pessimistic value operator of rough variable to transfer rough programming in to determinate one.

In this paper we suppose $0.5 < \alpha \leq 1$, according to the theorem, α –value of the rough variable ξ are $\xi_{sup}(\alpha)$ and $\xi_{inf}(\alpha)$ and $\xi_{inf}(\alpha) \geq \xi_{sup}(\alpha)$, it is denoted by $[\xi_{sup}(\alpha), \xi_{inf}(\alpha)]$.

3.1 Transfer rough model into deterministic model

Jiuping Xu,bin Liu,Desheng Wu[3],transferred the rough variables in model (1) into an interval programming under trust level α , therefore rough variables $\hat{x}_j = \left(\hat{x}_{ij} \dots \hat{x}_{mj} \right)^t > 0$ and

$\hat{y}_j = \left(\hat{y}_{ij} \dots \hat{y}_{mj} \right)^t > 0$ can be transformed to $[x_j^{sup(\alpha)}, x_j^{inf(\alpha)}]$ and $[y_j^{sup(\alpha)}, y_j^{inf(\alpha)}]$.

And with assumption $0.5 < \alpha \leq 1$ we have $\xi_{inf}(\alpha) \geq \xi_{sup}(\alpha)$. Now RDEA model (1) can be transformed in to following program:

$$\begin{aligned} \text{Max} \left(\sum_{i=1}^m S_i^- + \sum_{r=1}^s S_r^+ \right) & \quad (2) \\ \text{s.t} \quad \sum_{j=1}^n \lambda_j [x_{ij}^{sup(\alpha)}, x_{ij}^{inf(\alpha)}] & \leq [x_{io}^{sup(\alpha)}, x_{io}^{inf(\alpha)}] \quad i = 1, \dots, m \\ \sum_{j=1}^n \lambda_j [y_{rj}^{sup(\alpha)}, y_{rj}^{inf(\alpha)}] & \geq [y_{ro}^{sup(\alpha)}, y_{ro}^{inf(\alpha)}] \quad r = 1, \dots, s \\ \lambda_j & \geq 0, j = 1, \dots, n. \end{aligned}$$

In order to transform the interval programming (2) in to deterministic linear programming with α trust level we assume:

$$\begin{aligned} \frac{\min}{\text{inputs}} \text{ DMU}_o & \longrightarrow \max \quad \text{output} \\ \frac{\max}{\text{outputs}} \text{ DMU}_j & \longrightarrow \min \quad \text{output} \quad j \neq o \end{aligned}$$

Now interval programming (2) can be transformed in to programming (3) as following

$$\begin{aligned} \text{max} \left(\sum_{i=1}^m S_i^- + \sum_{r=1}^s S_r^+ \right) & \quad (3) \\ \text{s.t} \quad \sum_{j=1}^n \lambda_j x_{ij}^{inf(\alpha)} + \lambda_{jo} x_{jo}^{sup(\alpha)} + S_i^- & = x_{io}^{sup(\alpha)} \quad i = 1, \dots, m \\ \sum_{j=1}^n \lambda_j y_{rj}^{sup(\alpha)} + \lambda_{ro} y_{ro}^{inf(\alpha)} - S_r^+ & = y_{ro}^{inf(\alpha)} \quad r = 1, \dots, s \\ S_i^- \geq 0 \quad i=1, \dots, m, \quad S_r^+ \geq 0 \quad r=1, \dots, s, \quad \lambda_j & \geq 0, j=1, \dots, n. \end{aligned}$$

The variables in the model (3) are not rough, i.e. they are deterministic, and the model (3) is a linear program. In model (3) DMU_o is ARDEA-efficient if and only if $S_i^- = 0$ and $S_r^+ = 0$.

4 Practical application

In this section, we will evaluate the operation performance of the six DMUs with two inputs and one output (see Table 1) using the proposed ARDEA model.

Table 1. Input and output data of 6 DMUs

	X_1	X_2	Y
DMU ₁	([1650, 1775], [1545, 1867])	([520,560], [450,600])	6790
DMU ₂	([2040, 2240], [1915, 2460])	([560,600], [500,650])	8000
DMU ₃	([1980, 2080], [1700, 2180])	([620,720], [560,780])	6550
DMU ₄	([1760, 1840], [1650, 1900])	([710,730], [680,760])	5250
DMU ₅	([2120, 2210], [1920, 2300])	([565,585], [470,620])	8260
DMU ₆	([1940, 2010], [1880, 2100])	([620,685], [560,720])	6280

4.2 Performance evaluation

Using the transformation technique described in the previous section, we transform the rough variables in Table 1 into certain variables. Suppose the trust level $\alpha=0.9$, the corresponding programming model's solutions of DMUs are summarized in table2:

Table 2. The result for ARDEA model

DMU	DMU ₁	DMU ₂	DMU ₃	DMU ₄	DMU ₅	DMU ₆
ARDEA model	0.00	432.88	0.00	0.00	788.10	0.00

In Table 2, DMU₁, DMU₃, DMU₄ and DMU₆ are ARDEA-efficient, i.e. the summations of their slacks are zero.

5. Conclusion

In this paper, we have developed an ARDEA model with rough parameters. This model can be used to evaluate the performance of DMUs when they want to change their inputs and outputs non-radially. In the process of solving the ARDEA model, we used the α -optimistic value and α -pessimistic value of rough variable to transfer the rough model into deterministic liner programming. Finally we illustrate the proposed method by an example.

References

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