An Algorithm for the Anchor Points of the PPS of the CCR Model

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Abstract
Anchor DMUs are a new class in the general classification of Decision Making Units (DMUs) in Data Envelopment Analysis (DEA). An anchor DMU in DEA is an extreme-efficient DMU that defines the transition from the efficient frontier to the free-disposability part of the boundary of the Production Possibility Set (PPS). In this paper, the anchor points of the PPS of the CCR model are investigated. A basic definition of anchor point based on the supporting hyperplanes of the PPS of CCR model is provided. Then, by using a variant of super-efficiency models, the necessary and sufficient conditions for a DMU to be an anchor DMU are provided via some theorems. To illustrate the applicability of the proposed model, some numerical examples are finally given.

Keywords: Data Envelopment Analysis (DEA); Production Possibility Set (PPS), Efficient and inefficient frontier.

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1. Introduction
An anchor point in DEA is an extreme-efficient DMU lying on the intersection of some strongly and weakly efficient frontiers of the PPS. An anchor point is, therefore, an extreme-efficient DMU in which some inputs can increase and/or some outputs can decrease without passing through the interior of the PPS. Anchor points play a significant role in DEA theory and applications. The concept of anchor point was used in Thanassoulis and Allen [11] (1998) for the generation of unobserved DMUs in order to reduce appropriately the DEA-inefficient boundary of the PPS. Anchor points were first named and identified by Allen and Thanassoulis [2] (2004). They proposed a method for detecting anchor points of the constant returns to scale production possibility set (CRS-PPS) with one input and multiple outputs. However, their method is not applicable to multiple inputs and outputs. Thanassoulis et al. [11] (2012) proposed another approach to identify anchor points, using the radial efficiency scores and slack variables at the optimal solution of envelopment models. They extended the proposed approach in Allen and Thanassoulis [12] (2004) to the multiple inputs and outputs case in variable returns to scale production possibility set (VRS-PPS) in order to improve envelopment by means of unobserved DMUs. Bougnol and Dulá [3] (2009) defined the anchor point for the VRS-PPS. They provided a specialized procedure to identify anchor points based on their geometrical properties. Rouse [10] (2004) employed this idea in identifying prices for health care services. For more detail about the notion and applications of the anchor DMUs, see Bougnol [4] (2001) and Allen and Thanassoulis [2] (2004). Since the set of anchor DMUs is a subset of the set of extreme DMUs, the set of extreme DMUs must be obtained. For this aim, one can use the proposed algorithms in Charnes, Cooper and Thrall [6] (1991) as well as Dulá and López [7] (2006) among others. This paper provides a definition for the anchor point of the CRS-PPS. Subsequently, it utilizes a novel approach to identify the anchor points of the PPS of the CCR model in the multiple inputs and outputs case; through testing all CCR-efficient DMUs by a variant of super-efficiency models (see models (3) and (4), after eliminating the inefficient-CCR DMUs from the PPS). In fact, extreme and non-extreme CCR-efficient DMUs and anchor DMUs can be obtained, using models (3) and (4). An advantage of the proposed approach is in determining which inputs (outputs) of anchor DMUs can increase (decrease) without penetrating into the interior of the production possibility set. Another advantage of the proposed approach is in discovering the edges of the PPS on which the anchor DMUs lie; whereas the aforementioned methods are unable to do these two advantages. Some useful facts related to the properties of models (3) and (4) and the necessary and sufficient conditions for a DMU to be an anchor DMU are stated and proved. In addition, three numerical examples are provided.

2. Background
Consider a set of n DMUs which is associated with m inputs and s outputs. Particularly, each DMU_j = (X_j ; Y_j) (j = 1,... , n) consumes amount x_{ij} (> 0) of input i and produces amount y_{rj} (> 0) of output r. The production possibility set T; T = \{(X ; Y) | X \in E^m, Y \in E^s, X \geq 0, Y \geq 0\} is based on postulate sets which are presented with a brief explanation (see Banker et al. [13] (1984)). One of the DEA models to evaluate the relative efficiency of a set of DMUs is the CCR model, which is, proposed by Charnes et al. [5](1978). The (PPS) of the CCR model can be defined as follows (Charnes
In which $X_j$ and $Y_j$ are vectors of inputs and outputs of $DMU_j$, respectively.

We will employ a DMU classification based on the categories (i) CCR-inefficient (weak efficient and interior), (ii) non-extreme CCR-efficient, and (iii) extreme CCR-efficient. The three categories define the subsets $\mathcal{F}$, $\mathcal{E}$, and $\mathcal{I}$, respectively. These three subsets partition the set $J$. Any DMU in $E$; lies on the boundary (non-extreme) ray and any DMU in $E$; lies on the extreme ray of the PPS of the CCR model and named as extreme DMU. The set $E^*$ is also called the frame of $J$. The frames are important in DEA because the PPS of the DEA models are constructed by them and the exclusion each of them alters the shape of the PPS. The PPS of the CCR model is depicted in Figures (1) and (2). In Figure (1), $J = \{D_1, D_2, D_3, D_4\}$, $F = \{D_4\}$, $E=\{D_3\}$ and $E^*=\{D_1, D_2\}$. Also $D_1$ and $D_2$ are anchor DMUs.

The input-oriented CCR model, corresponds to $DMU_k, k \in J$, is given by:

$$\min \quad \theta - \varepsilon \left( \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \right)$$

subject to:

$$\sum_{j \in J} \lambda_j x_{ij} - s_r^- = y_{rk}, \quad r = 1, ..., s$$

$$\sum_{j \in J} \lambda_j x_{ij} + s_i^- = \theta x_{ik}, \quad i = 1, ..., m$$

$$\lambda_j \geq 0,$$

$$s_r^+ \geq 0,$$

$$s_i^- \geq 0,$$

$$\theta \quad \text{free}$$

Also, the output-oriented CCR model, corresponds to $DMU_k, k \in J$, is as follows:

$$\max \quad \varphi + \varepsilon \left( \sum_{i=1}^{m} t_i^- + \sum_{r=1}^{s} t_r^+ \right)$$

subject to:

$$\sum_{j \in J} \lambda_j y_{ij} - t_r^+ = \varphi y_{ik}, \quad r = 1, ..., s$$

$$\sum_{j \in J} \lambda_j y_{ij} + t_i^- = x_{ik}, \quad i = 1, ..., m$$

$$\lambda_j \geq 0,$$

$$t_r^+ \geq 0,$$

$$t_i^- \geq 0,$$

$$\varphi \quad \text{free}$$

Where $\varepsilon$ is non-Archimedean small and positive number and $s_i^-$, $s_r^+$, $t_i^-$, and $t_r^+$, $i = 1, ..., m$, $r = 1, ..., s$ are called slack variables belonging to $R^{ \geq 0}$. Note that $s_i^-$ and $t_i^-$ represent input excesses; also $s_r^+$ and $t_r^+$ represent output shortfalls. In here, $\theta$, $\varphi$ and $\lambda_j (\geq 0), j \in J$, are real numbers. Models (1) and (2) are called envelopment forms (with non-Archimedean number).

$DMU_k$ is said to be strongly efficient (CCR-efficient) if and only if for each optimal solutions, either (i) or (ii) happen:

(i) $\theta^* = 1$ and $(s^*, s^-) = (0, 0)$

(ii) $\varphi^* = 1$ and $(t^*, t^-) = (0, 0)$

$DMU_k$ is said to be weak efficient if and only if for some optimal solutions, either (v) or (iv) happen:

(v) $\theta^* = 1$ and $(s^*, s^-) \neq (0, 0)$

(iv) $\varphi^* = 1$ and $(t^*, t^-) \neq (0, 0)$

Note that if $\theta^* < 1$ and $\varphi^* > 1$ then $DMU_k$ is an interior point of the PPS.
Each inefficient and weak efficient DMU in the CCR model is said to be a CCR-inefficient DMU. Efficient Frontier is the set of all points (real or virtual DMUs) with efficiency score is equal to unity. Efficient frontier is divided into two categories:

i) Strong efficient frontier is the set of all (real or virtual) strong efficient (CCR efficient) DMUs.

ii) Weak efficient frontier in which all its relative interior points (real or virtual DMUs) are weak efficient DMUs.

DMU \( \left( X_k, Y_k \right) \) is said to be non-dominated if and only if there is not any DMU \( \left( X, Y \right) \) (real or virtual) such that:

\[
(-X, Y) \geq (-X_k, Y_k) \quad \text{and} \quad (-X, Y) \neq (-X_k, Y_k)
\]

We use the following theorem in the next section.

**Theorem 1**: There does not exist any virtual DMU (a member of the PPS) that dominates an DEA-efficient DMU.

**Proof.** See H. Fukuyama et al. (2012).

In this paper, corresponding to each strong efficient DMU \( \left( X_j, Y_j \right) \), the virtual DMUs \( DMU_j = \left( x_j, \ldots, x_m, y_j, \ldots, y_n \right) \) and \( DMU_j^\text{q} = \left( x_j, \ldots, x_m, y_j, \ldots, y_n - \gamma, \ldots, y_n \right) \) in which \( \alpha, \gamma > 0 \), are called “Dominated Input Virtual” \( DIV_k^l \) and “Dominated Output Virtual” \( DOV_k^q \) DMUs, respectively. These virtual DMUs are either interior points of the PPS of the CCR model or lie on some weak efficient frontiers (see theorems 3 and 6 and the proof of theorem 4). In the latter case we call these virtual DMUs as “weak efficient virtual DMUs” or WEV DMUs, hereafter.

It is important to note that the WEV DMUs play an important role in identifying anchor points. Accordingly, this paper tries to find them. In figure 1, DMU \( D_1 = \left( x_{12}, x_{22}, y_{12} \right) \) is strong efficient DMU and \( DIV_1^2, D'_1 = \left( x_{12}, x_{22} + \alpha, y_{12} \right) \) is a WEV DMU. In addition, in figure 2, \( DOV_1^2 D''_1 \) and \( DOV_1^1 D''_1 \) corresponding to DMU \( D_1 \), are WEV DMUs.

The following definition introduces the anchor points of the PPS of the CCR model.

**Figure 1**: \( DIV_1^2 \) DMU \( D_2 \) is a WEV DMU and also model 3 corresponding to \( D_2 \) with \( l=1 \) is infeasible therefore, \( D_2 \) is an anchor point.
Definition. DMU \(_k \in E^*\) is an anchor DMU if it belongs to an unbounded face of the PPS of the CCR model.

Remark 1. By the above definition; DMU \(_k \in E^*\) is an anchor DMU if there exist some \(l\) (or \(q\)) so that \(\text{DIV}^l_k\) (or \(\text{DOV}^q_k\)) DMUs are WEV DMUs.

In figure 1, \(\text{DIV}^1_2\) DMU \(D_2^*\) is a WEV DMU and so, \(D_2^*\) is an anchor point.

Throughout this paper, we must assume that there are not any two strong efficient DMUs as \((x, y)\) and \((tx, ty)\) for all \(t > 0\) and \(t \neq 1\). Otherwise, one of them must be deleted.

3. Identifying the anchor DMUs of the PPS of the CCR model

In this section, the anchor DMUs of the PPS of the CCR model are defined in the following way.

First, each \(\text{DMU}_k\); \((k \in I)\), is evaluated by models (1) or (2). Then, we hold all CCR-efficient DMUs, and remove other DMUs. Suppose that the set of all CCR-efficient DMUs is denoted by \(E = E \cup E^*\). Corresponding to each \(\text{DMU}_k = (x_{ik}, \ldots, x_{mk}, y_{lk}, \ldots, y_{sk})\), \((k \in E^*)\), the following models are solved:

\[
\begin{align*}
\text{min} & \quad \theta^k_l \\
\text{s.t.} & \quad \sum_{j \in E^{*} - \{k\}} \lambda^k_j y_{ij} \geq y_{sk}, \\
& \quad r = 1, \ldots, s, \\
& \quad \sum_{j \in E^{*} - \{k\}} \lambda^k_j x_{ij} \leq x_{sk}, \\
& \quad i = 1, \ldots, m, \quad i \neq l, \\
& \quad \sum_{j \in E^{*} - \{k\}} \lambda^k_j x_{ij} \leq \theta^k_l x_{sk}, \\
& \quad \lambda^k_j \geq 0, \quad j \in E^{*} - \{k\} \\
& \quad \theta^k_l \quad \text{free} \quad l = 1, \ldots, m \\
\end{align*}
\]  \hspace{1cm} (3)

\[
\begin{align*}
\text{max} & \quad \varphi^k_q \\
\text{s.t.} & \quad \sum_{j \in E^{*} - \{k\}} \mu^k_j y_{ij} \geq y_{sk}, \\
& \quad r = 1, \ldots, s, \quad r \neq q, \\
& \quad \sum_{j \in E^{*} - \{k\}} \mu^k_j y_{iq} \geq \varphi^k_q y_{sk}, \\
& \quad \sum_{j \in E^{*} - \{k\}} \mu^k_j x_{ij} \leq x_{sk}, \\
& \quad i = 1, \ldots, m, \\
\end{align*}
\]  \hspace{1cm} (4)
\[ \mu_j^k \geq 0, \quad j \in E \setminus \{k\} \]
\[ \varphi_q^k \quad \text{free} \quad q = 1, \ldots, s \]

The following Theorems are held for models (3) and (4). The theorems 3-8 provide the necessary and sufficient conditions for a DMU to be an anchor DMU.

**Theorem 2**: In model (3) (or (4)), if for some \( l \) (or \( q \)), \( \theta_l^{k^*} > 1 \) (or, \( \varphi_q^{k^*} < 1 \)) or if for some \( l \) (or \( q \)), model (3) (or model (4)) is infeasible, then, DMU \( k \) is an extreme DMU and vice versa.

**Proof.** Suppose that \( \theta_l^{k^*} > 1 \). First, we show that DMU \( k \) is CCR-efficient. By contradiction, suppose that DMU \( k \) is CCR-inefficient. Let \( (\theta^*, S^+, S^-) \) be the optimal solution of model (1).

Two cases can occur:

(i) \( \theta^* = 1 \) and \( (S^+, S^-) \neq (0, 0) \)

(ii) \( \theta^* < 1 \)

in each case it can be shown that \( \theta_l^{k^*} \leq 1 \), a contradiction.

Now we show that DMU \( k \) is, in fact, an extreme CCR-efficient DMU. By contradiction suppose that DMU \( k \) is a non-extreme CCR-efficient. So, the following system has solution:

\[
\sum_{j \in E'} \lambda_j y_j = y_k,
\]
\[
\sum_{j \in E'} \lambda_j x_j = x_k,
\]
\[
\lambda_j \geq 0, \quad j \in E'
\]

Suppose that \( \lambda_j, \quad j \in E' \) is a solution of the above system. If \( \lambda_k = 0 \) then, \( (\theta_l^1 = 1, \lambda_j = \lambda_k, j \in E \setminus \{k\}) \) is a solution of model (3). Therefore, \( \theta_l^{k^*} \leq 1 \), a contradiction. On the other hand if \( \lambda_k \neq 0 \) system (5) can be rewritten as follows:

\[
\sum_{j \in E \setminus \{k\}} \lambda_j y_j = (1 - \lambda_k) y_k,
\]
\[
\sum_{j \in E \setminus \{k\}} \lambda_j x_j = (1 - \lambda_k) x_k
\]

By dividing both sides of the above equations by \((1 - \lambda_k)\); a solution of model (3) is obtained as \( (\theta_l^1 = 1, \lambda_j = \lambda_k / (1 - \lambda_k), j \in E \setminus \{k\}) \).

Therefore, \( \theta_l^{k^*} \leq 1 \), a contradiction. Thus, DMU \( k \) is an extreme CCR-efficient DMU. Now, suppose that for some \( l \), model (3) is infeasible. In the similar manner, it can be shown that DMU \( k \) is an extreme CCR-efficient DMU. Conversely, suppose that DMU \( k \) is extreme DMU and model (3) is feasible. We show that \( \theta_l^{k^*} \geq 1 \).

Consider the following problem corresponding to DMU \( k \):

\[
\min \quad \theta_l^{k^*}
\]
\[
s.t. \quad \sum_{j \in E'} \lambda_j^r y_{rj} - s_r^+ = y_{rk},
\]
\[
r = 1, \ldots, s
\]
\[
\sum_{j \in E'} \lambda_j x_{ij} + s_i^- = x_{ik},
\]
\[
i = 1, \ldots, m, \quad i \neq l
\]
\[
\lambda_j^r \geq 0, \quad j \in E'
\]
\[
s_r^+, s_i^- \geq 0, \quad r = 1, \ldots, s \quad i = 1, \ldots, m,
\]
\[
\theta_l^{k^*} \quad \text{free} \quad l = 1, \ldots, m
\]

Now suppose that \( \theta^*(=1), \theta_l^{k^*} \) and \( \theta_l^{k^*} \) are the optimal objective functions of the models (1), (6) and (3) with respect to DMU \( k \), respectively. It is not difficult to
show that $\theta^* \leq \theta_{l}^{k*,1} \leq \theta_{l}^{k*,2}$ Therefore, $\theta_{l}^{k*,1} \geq 1$. This completes the proof.

**Corollary:** In models (3) and (4), for each $l$ and $q$ $\theta_{l}^{k*,1} = \varphi_{q}^{k*,1} = 1$ if and only if $DMU_k$ is a non-extreme CCR-efficient $DMU$.

**Proof.** Omitted.

**Theorem 3:** In a single input case, corresponding to each $DMU_k = (x_{1k}, y_{ik}, \ldots, y_{sk})$ the $DIV_{l}^i$ $DMU_k' = (x_{ik} + \alpha, y_{ik}, \ldots, y_{sk})$ in which $\alpha > 0$, is an interior point of the PPS of the CCR model.

**Proof.** First, we add $DMU_k'$ to the PPS and then, evaluate its performance by the input and output-oriented CCR models (see models (1) and (2)). It is enough to show that $\theta^* < 1$ and $\varphi^* > 1$. Consider the input-oriented CCR model corresponding to virtual $DMU$ $DMU_k'$ as follows:

$$\begin{align*}
\min_{\theta} & \sum_{j \in E'} \lambda_j y_{ij} + \mu_k y_{ik} \\
& \sum_{j \in E'} \lambda_j x_{ij} + \mu_k (x_{ik} + \alpha) \leq \theta(x_{ik} + \alpha), \\
& \lambda_j \geq 0, \quad j \in E' \\
& \theta \geq 0 \\
& (\lambda_{j}, = 0 \ (j \neq k), \bar{\lambda}_k = 1, \bar{\mu}_k = 0, \theta = \frac{x_{ik}}{x_{ik} + \alpha} < 1)
\end{align*}$$

is a feasible solution of (7). Since model (7) has a minimization-type objective function, $\theta^* < 1$; where "***" is used to indicate optimality. In a similar manner, it can be shown that in output-oriented maximization problem, $\varphi^* > 1$. Therefore, $DMU_k$ is an interior point of the PPS. This completes the proof.

In Figure 2, corresponding to $DMU D_1 = (x_{11}, y_{11}, y_{21})$, the $DIV_{l}^i D_1' = (x_{11} + \alpha, y_{11}, y_{21})$ is an interior point of the PPS.

**Theorem 4:** In a multiple inputs case, if for some $l$, model (3) is infeasible, then extreme CCR-efficient $DMU_k'$ is an anchor DMU.

**Proof.** In view of Remark 1, we show that if for some $l$, model (3) is infeasible, then, the $DIV_{l}^i$ $DMU_k' = (x_{1j}, \ldots, x_{(i,j)}, x_{ij} + \alpha, x_{mj}, y_{ij}, \ldots, y_{ij})$ is a WEV DMU. For this aim, it can be shown in the performance evaluation of $DMU_k'$, using model (2); $\varphi^* = 1$. Consider model (2); corresponding to virtual $DMU$ $DMU_k'$ as follows (without $\epsilon$):

$$\begin{align*}
\max_{\varphi} & \varphi \\
& \sum_{j \in E'} \lambda_j y_{ij} + \mu_k y_{ik} \geq \varphi y_{ik}, \\
& \sum_{j \in E'} \lambda_j y_{ij} + \mu_k y_{ik} \leq \varphi y_{ik}, \\
& \lambda_j \geq 0, \quad j \in E' \\
& \varphi \geq 0 \\
& \varphi \ 	ext{free}
\end{align*}$$

By contradiction, suppose that $(\lambda_j^*(j \in E'), \mu_k^*, \varphi(> 1))$ is the optimal solution of (8). The constraints of model (8) can be written as follows:
\[
\sum_{j \notin E \setminus \{k\}} \lambda_j^* y_{j, r} > (1 - \lambda_k^* - \mu_k^*) y_{r, k},
\]
\[
r = 1, ..., s
\]
\[
\sum_{j \notin E \setminus \{k\}} \lambda_j^* x_{j, i} \leq (1 - \lambda_k^* - \mu_k^*) x_{i, k},
\]  
(9)
\[
i = 1, ..., m, \ s, \ i \neq l
\]
\[
\sum_{j \notin E \setminus \{k\}} \lambda_j^* x_{j, i} \leq (1 - \lambda_k^* - \mu_k^*) x_{i, k} + (1 - \mu_k^*) \alpha,
\]

From model (9), it is easy to show that \(1 - \lambda_k^* - \mu_k^* > 0\). Divide both sides of model (9) by \(1 - \lambda_k^* - \mu_k^* > 0\) and define \(\mu_j^* = \frac{\lambda_j^*}{1 - \lambda_k^* - \mu_k^*}, \ j \in E \setminus \{k\}\); so, model (9) becomes as follows:

\[
\sum_{j \notin E \setminus \{k\}} \bar{\mu}_j y_{j, r} > y_{r, k}, \ r = 1, ..., s
\]
\[
\sum_{j \notin E \setminus \{k\}} \bar{\mu}_j x_{j, i} \leq x_{i, k}, \ i = 1, ..., m, \ i \neq l
\]
\[
\sum_{j \notin E \setminus \{k\}} \bar{\mu}_j x_{j, i} \leq x_{i, k} + \beta
\]

in which \(\beta = \left(1 - \frac{\mu_k^*}{1 - \lambda_k^* - \mu_k^*}\right) \alpha\). Since \(\beta > 0\), there is \(\hat{\theta} > 0\); so that \(x_{ik} + \beta = \hat{\theta} x_{ik}\); therefore, the constraints of model (10) can be rewritten as follows:

\[
\sum_{j \notin E \setminus \{k\}} \bar{\mu}_j y_{j, r} > y_{r, k}, \ r = 1, ..., s
\]
\[
\sum_{j \notin E \setminus \{k\}} \bar{\mu}_j x_{j, i} \leq x_{i, k}, \ i = 1, ..., m, \ i \neq l
\]
\[
\sum_{j \notin E \setminus \{k\}} \bar{\mu}_j x_{j, i} \leq \hat{\theta} x_{i, k}
\]

So, \(\left(\bar{\mu}_j \ (j \in E \setminus \{k\}), \ \hat{\theta}\right)\) is a feasible solution for model (3); a contradiction.

This implies that \(\varphi^* = 1\) i.e. \(DMU'_k\) lies on the efficient frontier. Now, since \(DMU'_k\) is dominated by CCR-efficient \(DMU_k\), so, the \(DIV^l_k\) \(DMU'_k\) is a WEV DMU. Therefore, in view of Remark 1 \(DMU'_k\) is an anchor DMU. This completes the proof.

In Figure 1, model (3) corresponding to DMU \(D_2 = (x_{12}, x_{22}, y_{12})\), with \(\ell = 1\), is infeasible; so, DMU \(D_2\) is an anchor DMU. The following theorem is, in fact, the converse of Theorem 4.

**Theorem 5:** In multiple inputs case, if extreme CCR-efficiency DMU \(DMU'_k = (x_{ik}, ..., x_{nk}, y_{il}, ..., y_{nk} l = 1, ..., k\) is an anchor DMU and the \(DIV^l_k\) DMU is a WEV DMU; then model (3) is infeasible.

**Proof.** By contradiction, suppose that model (3) is feasible. The first constraint of the model (3) implies that the optimal solution of model (3) is bounded. Suppose that, \((\theta^*_k, \lambda^*_k \ (j \neq k))\) is an optimal solution of it. Note that the first constraint of model (3) is tight at optimality. We first show that \(\theta^*_k > 1\). By contradiction suppose \(\theta^*_k \leq 1\). If \(\theta^*_k < 1\) we have:

\[
\sum_{j \notin E \setminus \{k\}} \lambda_j^* x_{j, i} = \theta^*_k x_{ik} < x_{ik}
\]
\[
\sum_{j \notin E \setminus \{k\}} \lambda_j^* x_{j, i} \leq x_{ik},
\]
\[
i = 1, ..., m, \ i \neq l
\]
\[
\sum_{j \notin E \setminus \{k\}} \lambda_j^* y_{j, r} \geq y_{r, k},
\]
\[
r = 1, ..., s
\]

(11)

It shows that virtual DMU
Dominates the CCR-efficient \( DMU_k \), a contradiction (see Theorem 1). Now, if \( \Theta^{*}=1 \), we have:

\[
\begin{align*}
\sum_{j \in E \setminus \{k\}} \lambda_{j}^{k+} x_{ij} & , \ldots , \\
\sum_{j \in E \setminus \{k\}} \lambda_{j}^{k+} y_{ij} & , \ldots , \\
\lambda_{j}^{k} x_{mj} , \\
\sum_{j \in E \setminus \{k\}} \lambda_{j}^{k+} y_{1j} & , \ldots , \\
\sum_{j \in E \setminus \{k\}} \lambda_{j}^{k} y_{sj} , \ldots ,
\end{align*}
\]

At least one of the inequality constraints of (12) is a strict inequality, because, otherwise, the CCR-efficient \( DMU_k \), is not extreme DMU and so \( \Theta^{*}>1 \). Therefore, there exist \( \beta>0 \) so that \( \Theta_{ik}^{*} x_{ik} = x_{ik} + \beta \). This means that, the virtual DMU

\[
DMU_k^{'} = \left( x_{1k} , \ldots , x_{(l-1)k} , x_{ik} + \beta , x_{(l+1)k} , \ldots , x_{mk} , y_{1k} , \ldots , y_{sk} \right)
\]
is, in fact, an observed DMU belongs to the PPS of the CCR model. This is a contradiction because; the CCR-inefficient DMUs had been eliminated from the PPS of the CCR model. The proof is completed.

**Theorem 6:** In a single output case, for each \( DMU_k = (x_{1k} , \ldots , x_{mk} , y_{1k}) \) the DOV\(_k\) \( DMU_k = (x_{1k} , \ldots , x_{mk} , y_{1k} - \gamma) \) in which \( \gamma > 0 \) is an interior point of the PPS of the CCR model.

**Proof.** The proof is similar to the theorem 3 and so, the details are omitted.

In Figure 3, DOV\(_k\) \( DMU_k = (x_{11} , x_{12} , y_{11} - \gamma) \) is an interior point of the PPS.

**Theorem 7:** In multiple outputs case, if for some \( q \), model (4) is infeasible, then, CCR-efficient \( DMU_k \) is an anchor DMU.

**Proof.** The proof is similar to theorem 4 except that it can be shown that in the performance evaluation of \( DMU_k^{'} \) using model (1); \( \Theta^{'} = 1 \).

In Figure 2, model (4) corresponding to \( DMU_{D_1} = (x_{11} , y_{11}, y_{21}) \), with \( q = 2 \), is infeasible, so, the DOV\(_{D_1}^2 \) DMU is a \( D_{1}^{'} = (x_{11} , y_{11}, y_{21} - \gamma) \) WEV DMU.

The following theorem is, in fact, the converse of theorem 7.
Theorem 8: In multiple outputs case, let extreme CCR-efficiency DMU
\(DMU_k = (x_{1k}, ..., x_{nk}, y_{1k}, ..., y_{qk}, ..., y_{sk})\)
is an anchor DMU and the \(DOV_k^q\) DMU is a WEV DMU; then model (4) is infeasible.

Proof. The proof is similar to the theorem 5 except that by contradiction it must be assumed that the model (4) is feasible. So, we omit it.

Now, by theorem 2; all extreme DMUs of the PPS of the CCR model can be found. Also, by theorems 4 and 5; all anchor DMUs for which the \(DIV_k^l\) DMUs are WEV DMUs can be found and therefore, all anchor points of the PPS of the CCR model can be found.

Now we are in the position to put all together the ingredients of the method.

Summary of finding all anchor DMUs algorithm

- **Step 1.** Evaluate n DMUs with a suitable form of models (1) and (2). Hold all CCR-efficient DMUs and remove other DMUs. Put indices of these CCR-efficient DMUs in \(E'\).

- **Step 2.** Evaluate each DMU in \(E'\) with models (3) and (4). (Note that in the single input case we don’t use model (3) and in the single output case we don’t use model (4)).

- **Step 3.** If for some \(l\) (or \(q\)) the model (3) (or (4)) is infeasible, then, \(DMU_k^l\) is an anchor DMU and \(DIV_k^l\) (or \(DOV_k^q\)) \(DMU_k^l\) is WEV DMU.

- **Step 4.** If each DMU in \(E'\) are evaluated by models (3) and (4), stop. Otherwise, go to step 1.
4. Numerical Examples

Example 1 (Single output case)
Table 1 shows data for 4 DMUs with two inputs and one output. Using the CCR model (1), CCR-efficient DMUs are determined to be $D_1$, $D_2$ and $D_3$. So, $E' = \{1, 2, 3\}$. Remove CCR-inefficient DMU $D_3$ from PPS and solve model (3) corresponding to CCR-efficient DMUs $D_1$, $D_2$ and $D_4$. The following results are yielded:

By theorem 2, DMUs $D_1$, $D_2$ and $D_4$ lie on the extreme rays of the PPS. Model (3) corresponding to DMU $D_1$ with $l = 2$ and DMU $D_4$ with $l = 1$ is infeasible. So, by theorem 4, $D_1$ is an anchor DMU and $DIV_1^2$ DMU $D_1 = (2, 3 + \alpha, 7, 4)$ is a WEV DMU. Using theorems 4 and 7 and the information of table 3, all $DMU_k$, $k \in E'$ are anchor DMUs.

Example 2 (Multiple outputs and inputs case)
Table 2 shows data for 5 DMUs with two inputs and two outputs. Running model (1) (or (2)) shows that $D_1$, $D_2$ and $D_4$ are CCR-efficient and other DMUs are CCR-inefficient. So, $E' = \{1, 2, 4\}$. By applying models (3) and (4) to each $DMU_k$, $k \in E'$ the results reported in table 3 are obtained. In table 3, “INFES” and “FES” denotes “infeasible” and “feasible”, respectively. For instance, “INFES” in the first row and the second column means that model (3), corresponding to DMU $D_1$ with $l = 2$, is infeasible. So, by theorem 4, $D_1$ is an anchor DMU and $DIV_1^2$ DMU $D_1 = (2, 3 + \alpha, 7, 4)$ is a WEV DMU. Using theorems 4 and 7 and the information of table 3, all $DMU_k$, $k \in E'$ are anchor DMUs.

Example 3 (Real world data)
We evaluated the data of 20 branches of a bank in Iran using the proposed method. The data was previously analyzed by Amirteimoori et al. (2005), (see table (4)). Running the DEA model (1) (or (2)) resulted in $E' = \{1, 4, 7, 12, 15, 17, 20\}$. Using the proposed method, all DMUs in $E'$ are found to be anchor DMUs. Also $DIV_1^{1,2}$, $DIV_4^{1,2,3}$, $DIV_1^{1,2,3}$, $DIV_2^2$, $DIV_3^{1,2,3}$, $DIV_4^{1,2,3}$, $DIV_5^{1,2,3}$, $DIV_2^{1,2,3}$, $DIV_3^{2,3}$ and $DOV_1^2$, $DOV_2^2$, $DOV_3^2$, $DOV_4^2$.
DOV\textsuperscript{1,2}, DOV\textsuperscript{1,3}, DOV\textsuperscript{12}, DOV\textsuperscript{15}, DOV\textsuperscript{1,3}, DOV\textsuperscript{12}, DOV\textsuperscript{20} DMUs are WEV DMUs. For instance, DIV\textsuperscript{12,3} means that, DMU\textsubscript{12} is an anchor point and the first, second and the third inputs of DMU\textsubscript{12} can be increased without penetrating the interior of the PPS. Also, DOV\textsuperscript{1,3} means that, DMU\textsubscript{7} is an anchor point and the first and third outputs of DMU\textsubscript{7} can decrease without penetrating the interior of the PPS.

Table 3: Example 2. The results of evaluation CCR-efficient DMUs by models (3) and (4).

<table>
<thead>
<tr>
<th>DMU</th>
<th>l</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>D\textsubscript{1}</td>
<td>FES</td>
<td>INFES</td>
</tr>
<tr>
<td>D\textsubscript{2}</td>
<td>INFES</td>
<td>INFES</td>
</tr>
<tr>
<td>D\textsubscript{3}</td>
<td>INFES</td>
<td>FES</td>
</tr>
</tbody>
</table>

Table 4: Example 3. DMUs’ data (extracted from [Amirteimoori et al. (2005), p. 689]).

<table>
<thead>
<tr>
<th>Branch</th>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Staff</td>
<td>Deposits</td>
</tr>
<tr>
<td></td>
<td>Computer terminals</td>
<td>Space m\textsuperscript{2}</td>
</tr>
<tr>
<td>1</td>
<td>0.9503</td>
<td>0.70</td>
</tr>
<tr>
<td>2</td>
<td>0.7962</td>
<td>0.60</td>
</tr>
<tr>
<td>3</td>
<td>0.7982</td>
<td>0.75</td>
</tr>
<tr>
<td>4</td>
<td>0.8651</td>
<td>0.55</td>
</tr>
<tr>
<td>5</td>
<td>0.8151</td>
<td>0.85</td>
</tr>
<tr>
<td>6</td>
<td>0.8416</td>
<td>0.65</td>
</tr>
<tr>
<td>7</td>
<td>0.7189</td>
<td>0.60</td>
</tr>
<tr>
<td>8</td>
<td>0.7853</td>
<td>0.75</td>
</tr>
<tr>
<td>9</td>
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</tr>
<tr>
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<tr>
<td>11</td>
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</tr>
<tr>
<td>12</td>
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<td>0.65</td>
</tr>
<tr>
<td>13</td>
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<td>0.85</td>
</tr>
<tr>
<td>14</td>
<td>0.9763</td>
<td>0.80</td>
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<tr>
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<td>0.95</td>
</tr>
<tr>
<td>16</td>
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</tr>
<tr>
<td>17</td>
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</tr>
<tr>
<td>18</td>
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</tr>
<tr>
<td>20</td>
<td>0.5827</td>
<td>0.55</td>
</tr>
</tbody>
</table>

5. Conclusions
Anchor points play an important role in DEA theory and application. They delineate the efficient frontier from the free-disposability portion of the PPS frontier. Their identification has several notable DEA applications such as the construction of "unobserved" DMUs in order to reduce appropriately the DEA-inefficient boundary of the PPS. This paper proposed a method for finding all anchor DMUs of the PPS of the CCR model using two super-efficiency models (see models (3) and (4)). The necessary and sufficient conditions for a DMU to be an anchor DMU were stated and proved. The advantage of our approach is in determining inputs (outputs) of anchor
DMUs that can increase (decrease) without penetrating into the interior of the production possibility set. Another advantage of our approach is in clarifying the edges of the PPS on which anchor DMUs lie; whereas the aforesaid methods are unable to do these two advantages. Initial studies had shown that our approach can also be applied to BCC model. We suggest a deeper analysis in this subject as future works. Finally, the GAMs software has been utilized to run the models (3) and (4).
References


